

Last time: $n \geq 1$

$$D_n = \langle x, y \mid x^n = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$$

Classify subgps of O_2 that are finite:
 either $\cong C_n$
 $\cong D_n$

Thm: The gp D_n has order $2n$, and $\exists$ injective homom $D_n \hookrightarrow O_2$.

Lemma: Suppose $S = \{a, b, c, d, \dots\}$ a set, and $\mathcal{F}_S$ denotes the free gp on S . Suppose moreover G' is a gp, and have a collection of elts $\{g'_s\}_{s \in S} \in G'$. Then $\exists$ a unique gp homom $\varphi: \mathcal{F}_S \rightarrow G'$ with $\varphi(s) = g'_s \ \forall s \in S$.

Ex $S = \{x, y\}$ $G' = S_3 = \langle x, y \mid x^3 = 1, y^2 = 1 \rangle$

Then $\mathcal{F}_{S=\{x,y\}} \rightarrow G' = S_3$

$$\begin{array}{ccc} x & \longmapsto & x \\ y & \longmapsto & y \end{array}$$

$$x^{-3} y x^5 y^3 \longmapsto x^{-3} y x^5 y^3$$

Pf of Thm

First step $|D_n| \leq 2n$

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Second step: $\varphi: D_n \rightarrow O_2$ and prove $\varphi(D_n) \geq 2n$

$$\Rightarrow 2n \geq |D_n| \geq |\varphi(D_n)| \geq 2n$$

$$\Rightarrow |D_n| = 2n \text{ and } \varphi \text{ is injective}$$

Step 1 Can use the rels $(x^n=1, y^2=1, yxy^{-1}=x^{-1})$

to put any word in x and y into the form

$$x^j \quad \text{or} \quad x^j y$$
$$0 \leq j \leq n-1 \quad \quad 0 \leq j \leq n-1$$

$$\{1, x, x^2, \dots, x^{n-1}, y, xy, \dots, x^{n-1}y\} \subset 2n \text{ elts}$$

PF: use $yx = x^{-1}y$ to move all the y 's in the word to the right

• Then $x^n=1, y^2=1$ to see that the power of x is between 0 and $n-1$ and the power of y is 0 or 1

$$\Rightarrow |D_n| \leq 2n.$$

$$\text{Define } X = \rho_{\frac{2\pi}{n}} = \begin{pmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix} \in O_2$$

$$Y = \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O_2$$

Lemma on free grps $\exists \varphi: \tilde{F}_{2n+1} \rightarrow O_2, \varphi \text{ homom.}$

Lemma on free gps $\exists \varphi: \mathcal{F}_{\{X,Y\}} \longrightarrow O_2$, φ homom.

$$\begin{array}{ccc} X & \xrightarrow{\quad} & X \\ Y & \xrightarrow{\quad} & Y \end{array}$$

Note: $X^n = 1$, $Y^2 = 1$, $YXY^{-1} = \rho_{-\frac{2\pi}{n}} = X^{-1}$

$\Rightarrow X^n \in \ker \varphi$, $Y^2 \in \ker \varphi$, $YXY^{-1}X \in \ker \varphi$

$\Rightarrow \varphi$ induces $\bar{\varphi}: \mathcal{F}_{\{X,Y\}} / \langle X^n, Y^2, YXY^{-1}X \rangle \longrightarrow O_2$

$\Downarrow$

$\mathcal{F}_{\{X,Y\}} / (\ker \varphi)$

$\ker \varphi \supseteq \langle X^n, Y^2, YXY^{-1}X \rangle$

smallest normal subgroup containing these elts

$2n \geq |D_n| \geq |\varphi(D_n)| = \langle X, Y \rangle \leq O_2$

subgp of O_2 gen'd by X, Y .

Claim: $\langle X, Y \rangle$ has size at least $2n$

$\varphi(D_n) = \{1, X, X^2, \dots, X^{n-1}, Y, XY, \dots, X^{n-1}Y\}$

Pfi

$\nearrow$ These $2n$ elts are distinct.

Pf:

These $2n$ elts are distinct.

Indeed, the X^j are distinct from the X^k as seen by taking determinants.

The X^j, X^k for $j \neq k$ in $\{0, 1, \dots, n-1\}$ are distinct because X has order n .

Cor of Prop: $\bar{\varphi}: D_n \longrightarrow O_2$
 $x \longmapsto p_{2\pi/n}$

$$y \longmapsto r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is an injective gp homom.

Propⁿ $D_3 \cong S_3$.

Pf: $D_3 = \langle x, y \mid x^3=1, y^2=1 \rangle \cong S_3 = \langle x, y \mid x^3=1, y^2=1 \rangle$ $\square$

For $n \geq 3$, $D_n \not\cong S_n$
 $\underbrace{\quad}_n \quad \underbrace{\quad}_{n!}$

Thm Suppose $G \leq O_2$ is a finite subgp.

a) If $G \leq SO_2$, then $G \cong C_n$ for some n

b) If $G \not\leq SO_2$, then $G \cong D_n$ for some n .

Defⁿ $\Gamma \leq (\mathbb{R}, +)$ is a subgp. Γ is said to be discrete

Defⁿ $\Gamma \leq (\mathbb{R}, +)$ is a subgp. Γ is said to be discrete if $\exists \varepsilon > 0$ so that $\forall \gamma \in \Gamma, \gamma \neq 0$ then $|\gamma| \geq \varepsilon$.

Lemma: ^{Suppose} $\Gamma \leq (\mathbb{R}, +)$ is discrete subgp. Then $\exists a > 0$ so that $\Gamma = \mathbb{Z}a$ or $\Gamma = \{0\}$.

Pf: If $\Gamma = \{0\}$ then done.

Otherwise, $\exists a' \in \Gamma, a' \neq 0 \Rightarrow -a' \in \Gamma$
 $\Rightarrow$ WLOG can assume $a' > 0$.

Claim: $\exists$ finitely many x in Γ with $0 \leq x \leq a'$.

Pf: There exists $\varepsilon > 0$ so that if $b, c \in \Gamma, b \neq c$ then $|b - c| \geq \varepsilon$. $\Rightarrow$ In any bounded interval of $\mathbb{R}$ there's at most finitely many elts of Γ . In particular, finitely many in $[0, a']$.

Claim $\Rightarrow \exists$ a smallest positive elt $a \in \Gamma$.

$\Rightarrow \mathbb{Z}a \subseteq \Gamma$ (clear)

Claim: $\Gamma \subseteq \mathbb{Z}a$

Pf: Suppose $b \in \Gamma \Rightarrow \exists r \in \mathbb{R}$ with $b = ra$.

$$r = m + r_0 \quad \text{for } m \in \mathbb{Z} \text{ and } 0 \leq r_0 < 1$$

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(eg. $-3.7 = -4 + .3$)

$$b = ra = (m + r_0)a = ma + r_0a$$

$$\Rightarrow \underbrace{b - ma}_{\in \Gamma} = r_0a \quad 0 \leq r_0a < a$$

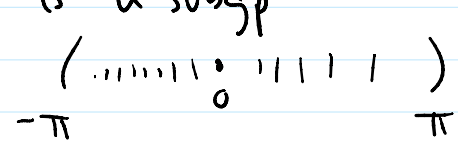
$$\Rightarrow r_0a = 0.$$

$$\Rightarrow b = ma \checkmark.$$

Pf of Thm

a) $G \leq SO_2$. Define

$$\Gamma = \left\{ \alpha \in \mathbb{R} : \underbrace{\rho_\alpha}_{\text{rotation by } \alpha} \in G \right\}.$$

- $\Gamma \leq (\mathbb{R}, +)$ is a subgroup
- Γ is discrete  because G is finite
- $2\pi \in \Gamma$ because $\rho_{2\pi} = 1$.

Lemma: $\Gamma = \mathbb{Z}a$ for some $a > 0$.

$$2\pi = n \cdot a \quad \text{for some int. } n$$

$$\Rightarrow a = \frac{2\pi}{n} \quad \text{for some } n.$$

$$\Rightarrow G = C_n = \langle \rho_{\frac{2\pi}{n}} \rangle \leq SO_2 \quad \checkmark.$$

$$\Rightarrow G = C_n = \langle p_{2\pi/n} \rangle \subseteq SO_2 \quad \checkmark$$

Pf of b. Denote by $H \subseteq G$ the subgroup consisting of rotations,

$$\text{a. } H = G \cap SO_2.$$

• Assumption: $G \not\subseteq SO_2$. Let r_g be in G but not in SO_2 .

r_g is conjugate inside O_2 to $r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\Rightarrow$ By conjugating G inside of O_2 , can assume $r \in G$.

• H , by part a, is $\triangleq C_n = \langle p_{2\pi/n} \rangle \subseteq SO_2$.

TL $p \in G$, then either

$$\bullet \det(p) = 1 \Rightarrow p \in H.$$

$$\bullet \det(p) = -1 \Rightarrow p = hr \text{ for some } h \in H$$

(consider pr^{-1} , which has $\det = 1$)

$$\Rightarrow p \in \bar{\varphi}(D_n) = \langle p_{2\pi/n}, r \rangle$$

$$\Rightarrow G = \bar{\varphi}(D_n) = \langle p_{2\pi/n}, r \rangle \cong D_n.$$